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A NOTE ON ACCURATE ESTIMATIONS OF THE LOCAL TRUNCATION ERROR OF POLYNOMIAL METHODS FOR DIFFERENTIAL EQUATIONS

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Abstract—We give sharp error estimations for the local truncation error of polynomial methods for the approximate solution of initial value problems. Our analysis is developed for second order differential equations with polynomial coefficients using the Tau Method as an analytical tool. Our estimates apply to approximations derived with the latter, with collocation and with techniques based on series expansions. An example on collocation is given, to illustrate the use of our estimates.

1. INTRODUCTION

Khajah and Ortiz [1] have recently proposed new techniques for estimating upper and lower bounds for the local truncation error of the Tau Method (see [2]). Namasivayam and Ortiz [3] discussed the case of equations with constant coefficients.

In this note, we consider differential equations with variable coefficients and give a sharp estimation formula which applies to any point of the interval of approximation.

In separate papers, we have shown (see [4, 5]) that a family of numerical techniques, which includes Galerkin's and the collocation methods, can be easily simulated in terms of the Tau Method with appropriate perturbation terms.

Our results are deduced for the Tau Method but apply to the larger class of numerical methods mentioned above. In an example, we show how it can be applied to collocation.

Let us consider the initial value problem defined by the second order differential operator D:

$$Dy(x) := y''(x) + B(x)y'(x) + C(x)y(x) = f(x);\ x \in [-1,1] \tag{1}$$
$$y(-1) = \gamma_1, \quad y'(-1) = \gamma_2; \qquad \{\gamma_1, \gamma_2\} \subset \mathbb{R}, \tag{2}$$

where the coefficients $B(x)$, $C(x)$ and $f(x)$ are some given polynomials.

2. DERIVATION OF AN ESTIMATION FORMULA

Let $G(x,t)$ be the Green's function of differential equation (1) associated with homogeneous initial conditions. Let us introduce the following polynomial functions:

$$\begin{array}{lll} B_0(x) = 0, & B_1(x) = -1, & B_2(x) = B(x), \\ C_0(x) = -1, & C_1(x) = 0, & C_2(x) = C(x) \end{array}$$

and, for $k > 2$, let

$$B_{k+1}(x) = B_k'(x) + C_k(x) - B_2(x)B_k(x) \tag{3}$$
$$C_{k+1}(x) = C_k'(x) - C_2(x)B_k(x). \tag{4}$$

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We need the following two lemmas:

LEMMA 1. *The following power series expansion holds true:*

$$G(x,t) = \sum_{k=0}^{\infty} -\frac{1}{k!} B_k(t)(x-t)^k \tag{5}$$

The proof of this lemma is given in [6].

Let h be a nonzero positive real number and n be a nonzero positive integer. Let $T_k^{[h]}(z)$ designate the Chebyshev polynomial of degree k defined on interval $[0,h]$. Let us define the following vectors and arrays:

(i) $\underline{\underline{z}} := (1, z, \ldots, z^n)^T$,

(ii) $\underline{\underline{T}}^{[h]}(z) := (T_0^{[h]}(z), T_1^{[h]}(z), \ldots, T_n^{[h]}(z))^T$

(iii) $\Delta_{[0,1]}$ is the transformation matrix taking the basis $\{x^k; k \in \mathbb{N}\}$ into the basis of *shifted* Chebyshev polynomials $\{T_k^*(x); k \in \mathbb{N}\}$, defined on interval $[0,1]$.

We prove now the following result:

LEMMA 2. *For all $z \in [0,h]$, we have*

$$\underline{\underline{z}} = \operatorname{diag}\{1, h, \ldots, h^n\} \Delta_{[0,1]}^{-1} \underline{\underline{T}}^{[h]}(z) \equiv \Delta_h \underline{\underline{T}}^{[h]}(z) \tag{6}$$

PROOF. If $T_j^*(z) := \sum_{i=0}^{j} c_i^j z^i$, then, for any given $h > 0$, we have

$$T_j^{[h]}(z) = \sum_{i=0}^{j} \frac{c_i^j}{h^i} z^i.$$

In matricial notation,

$$\underline{\underline{T}}^{[h]}(z) := \begin{pmatrix} T_0^{[h]}(z) \\ T_1^{[h]}(z) \\ T_2^{[h]}(z) \\ \vdots \\ T_n^{[h]}(z) \end{pmatrix} = \begin{pmatrix} c_0^0 & 0 & \cdots & & & \\ c_0^1 & \frac{c_1^1}{h} & 0 & \cdots & & \\ c_0^2 & \frac{c_1^2}{h} & \frac{c_2^2}{h^2} & 0 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ c_0^n & \frac{c_1^n}{h} & \frac{c_2^n}{h^2} & \cdots & \frac{c_n^n}{h^n} & \end{pmatrix} \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \\ z^n \end{pmatrix} \equiv \Delta_{[0,1]} \operatorname{diag}\{1, h^{-1}, \ldots, h^{-n}\} \underline{\underline{z}}.$$

Thus,

$$\underline{\underline{z}} = \operatorname{diag}\{1, h, \ldots, h^n\} \Delta_{[0,1]}^{-1} \underline{\underline{T}}^{[h]}(z). \qquad \blacksquare$$

Let us assume that $y_N(x)$ is a polynomial approximation of $y(x)$ obtained by the Tau Method applied to problem (1)–(2) in the interval $[-1,1]$ using a perturbation term $H_N(x)$. Let $e(x) := y(x) - y_N(x)$ be the error function. This leads us to the following theorem.

THEOREM 1. *For all $x \in [-1,1]$*

$$e(x) = -\sum_{k=0}^{\infty} \left[\sum_{j=k}^{\infty} \int_{-1}^{x} B_j(t) \frac{a_k^j}{j!} T_k^{[x+1]}(x-t) H_N(t)\, dt \right] \tag{7}$$

where the functions $\{B_i(x); i \in \mathbb{N}\}$ and $\{C_i(x); i \in \mathbb{N}\}$ are given by (3)–(4).

Proof. Let $x \in [-1,1]$ and $t \in [-1,x]$. Then $z := x - t \in [0, x+1]$. Setting $h = x+1$, from Lemma 2, we deduce that for all $z \in [0, x+1]$ and all $k \geq 0$

$$z^k = \sum_{j=0}^{k} a_j^k T_j^{[x+1]}(z), \tag{8}$$

where (a_j^k) are the entries of Δ_h. From Lemma 1, we have

$$G(x,t) = \sum_{k=0}^{\infty} \frac{B_k(t)}{k!} z^k; \qquad z = x - t. \tag{9}$$

Inserting (8) in the latter

$$G(x,t) = \sum_{k=0}^{\infty} \frac{B_k(t)}{k!} \left[\sum_{j=0}^{k} a_j^k T_j^{[x+1]}(z) \right] = \sum_{k=0}^{\infty} \left[\sum_{j=k}^{\infty} \frac{B_j(t)}{j!} a_k^j \right] T_k^{[x+1]}(z).$$

Therefore,

$$\begin{aligned} e(x) \equiv -\int_{-1}^{x} G(x,t) H_N(t)\,dt &= -\int_{-1}^{x} \left\{ \sum_{k=0}^{\infty} \left[\sum_{j=k}^{\infty} \frac{B_j(t)}{j!} a_k^j \right] T_k^{[x+1]}(x-t) \right\} H_N(t)\,dt \\ &= -\sum_{k=0}^{\infty} \left[\sum_{j=k}^{\infty} \int_{-1}^{x} \frac{B_j(t)}{j!} a_k^j T_k^{[x+1]}(x-t) H_N(t)\,dt \right]. \end{aligned}$$

∎

Definition 1. *For any* $(m,n) \in \mathbb{N} \times \mathbb{N}$

$$e_{m,n}(x) := -\sum_{k=0}^{m} \left[\sum_{j=k}^{n} \int_{-1}^{x} B_j(t) \frac{a_k^j}{j!} T_k^{[x+1]}(x-t) H_N(t)\,dt \right] \tag{10}$$

will be called the **estimated error function** *of order* (m,n).

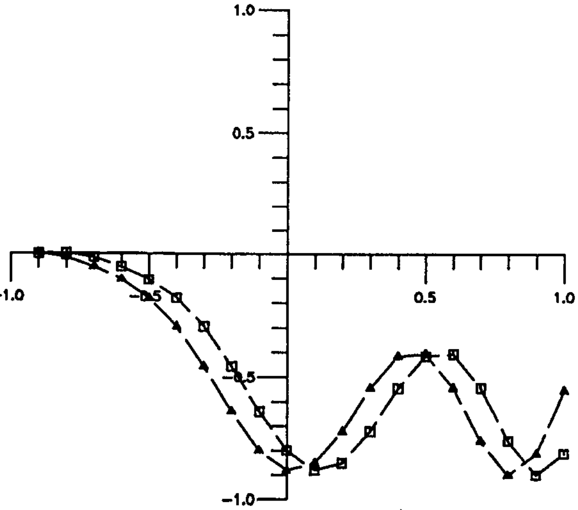

Figure 1. Approximation of IVP (11) by the global Tau Method: This figure shows the exact (◇) and estimated (◁) function errors at some selected points. Normalization factor is equal to 0.3E+3.

Table 1. We solve problem (11) by the global Tau Method: Estimated function and derivative errors at points $x_i = \pm i/10$, $i = 0(1)10$, were computed using (10) with $(m,n) = (10,100)$. Differences between exact and estimated values are also given.

mesh	Function		Derivative	
i	E_x :=Exact $e_i(x_i)$ E_s :=Estim. $e_i(x_i)$	$E_x - E_s$	E'_x :=Exact $e'_i(x_i)$ E'_s :=Estim. $e'_i(x_i)$	$E'_x - E'_s$
2	−4.7430728505521E-5 −4.7430728505583E-5	6.181E-17	−1.0534160168125E-3 −1.0534160168130E-3	5.029E-16
4	−3.5274661147833E-4 −3.5274661147838E-4	5.426E-17	−2.0158616466996E-3 −2.0158616466995E-3	−1.821E-16
6	−9.87664386303E-4 −9.87664386302E-4	−1.287E-15	−4.6474003486E-3 −4.6474003487E-3	1.364E-13
8	−2.13384439979E-3 −2.13384439973E-3	−5.993E-14	−6.003441979E-3 −6.003441977E-3	−2.360E-12
10	−2.9335961747E-3 −2.9335961742E-3	−5.437E-13	−9.5054515E-4 −9.5054522E-4	6.508E-11
12	−2.400742566E-3 −2.400742560E-3	−6.590E-12	5.5290155E-3 5.5290153E-3	1.744E-10
14	−1.37831151E-3 −1.37831160E-3	9.169E-11	2.676208E-3 2.676205E-3	3.136E-9
16	−1.810822E-3 −1.810824E-3	1.681E-9	−6.5653761E-3 −6.5654060E-3	2.948E-8
18	−2.997560E-3 −2.997562E-3	3.764E-9	−1.45080E-3 −1.45076E-3	−4.096E-8
20	−1.84423E-3 −1.84419E-3	−3.679E-8	6.9676E-3 6.9683E-3	−6.728E-7

3. APPLICATION TO THE COLLOCATION METHOD

Let us consider the linear second order differential equation with variable coefficients

$$\begin{aligned} &Dy := y''(x) + 4xy'(x) + (4x^2+2)\,y = 0, \qquad x \in [-1,1] \\ &y(-1) = 0, \quad y'(-1) = \exp(-1), \end{aligned} \tag{11}$$

the exact solution of which is $y(x) = \exp(-x^2) + x\exp(-x^2)$. We want to solve (11) by collocation at the zeros of Chebyshev polynomial $T_N(x)$, $N \in \mathbb{N}$.

Using the recursive formulation of the Tau Method we should choose the perturbation

$$H_N(x) = (\tau_0 + \tau_1 x + \tau_2 x^2 + \tau_3 x^3)\,T_N(x).$$

Errors in function and derivative are given in Table 1. They were estimated using (10) with $m = 10$ and $n = 100$. We remark that not only these values are oscillating but so does the difference between the exact and approximate error functions. This oscillatory behavior is due to the fact that we have truncated the Chebsyshev series expansion (7), which represents the exact error.

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